Fundamental Problems and Recent Progresses of Quantized Systems: From Parameter Identification to Adaptive Control

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Contents

- Why do we consider quantized systems
- What are the fundamental problems
- What are the recent progresses:
  identification, adaptive control, consensus
- What are the problems worth studying

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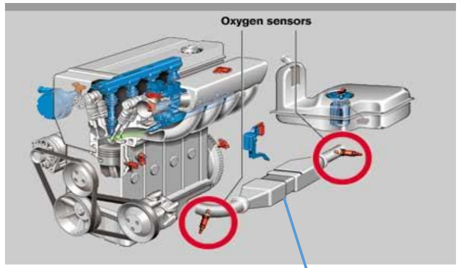
Why do we consider quantized systems

- ✓ Need of practical systems
- ✓ Need of the development of control theory

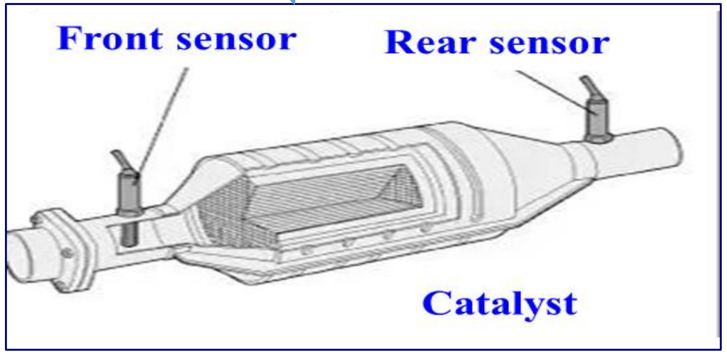
Quantized systems: binary-valued data

Internal combustion engine air-fuel ratio (AFR) control

Only when engine AFR is 14.7, the toxic tail gas can be effectively cleaned by the catalyst. Thus, accurate control of AFR is very important.



- Front oxygen sensor (usually wide range type): detects the air-fuel ratio
- Rear oxygen sensor (wide range or switch type): detect the conversion efficiency of the catalyst & the operation of upstream oxygen sensor.



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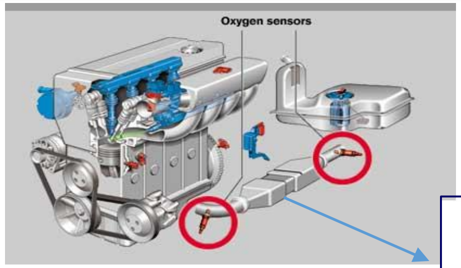

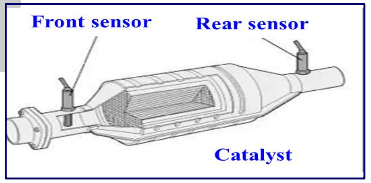

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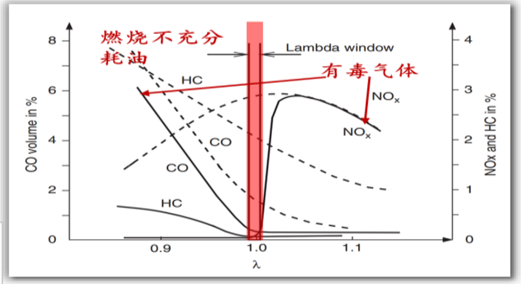
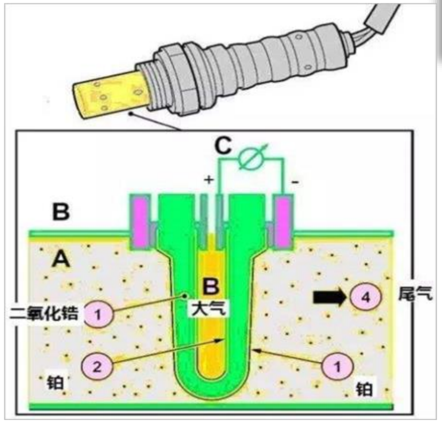
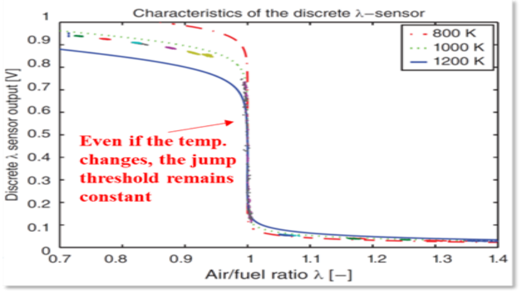


Measuring principle: under high temperature and platinum Catalysis, the oxygen concentration difference on both sides of zirconia increases, the corresponding electromotive force difference increases.

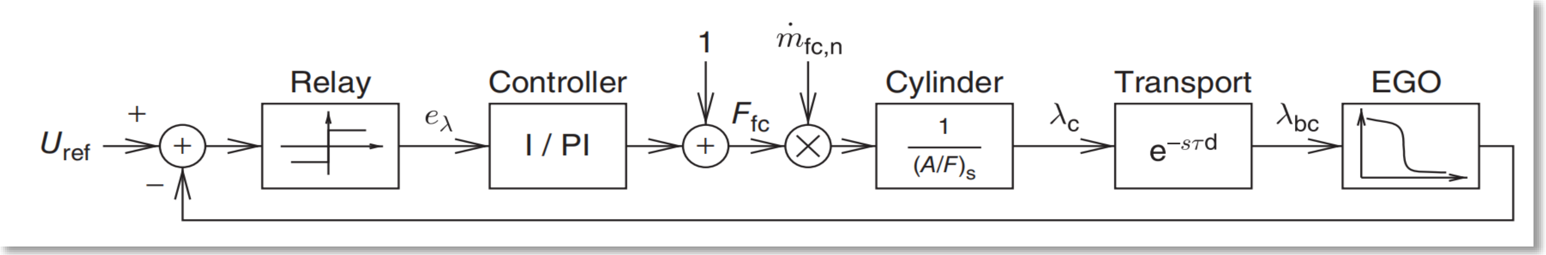




Quantized systems: binary-valued data

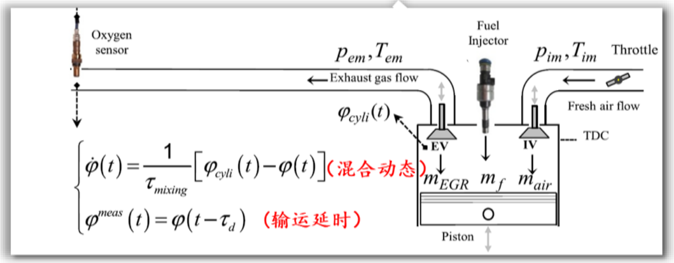
■ Internal combustion engine air-fuel ratio control

Kang Song, Tianyuan Hao, and Hui Xie. "Disturbance rejection control of air–fuel ratio with transport-delay in engines." *Control Engineering Practice* 79 (2018): 36-49.



$$F_{fc} = (1 + \Delta_{fc})$$

$$\Delta_{fc} = K_I \int e_\lambda dt$$



- **Transport-delay:**

$$\tau_{d,1} \approx \hat{\tau}_{d,1} = \frac{120}{N_{Eng}}$$

$$\tau_{d,2} \approx \hat{\tau}_{d,2} = \frac{V_{em}}{C_{ex}}$$

N_{eng}: 发动机转速
V_{em} : 排气管容积
C_{ex} : 排气体积流率

- **Mixed dynamics:**

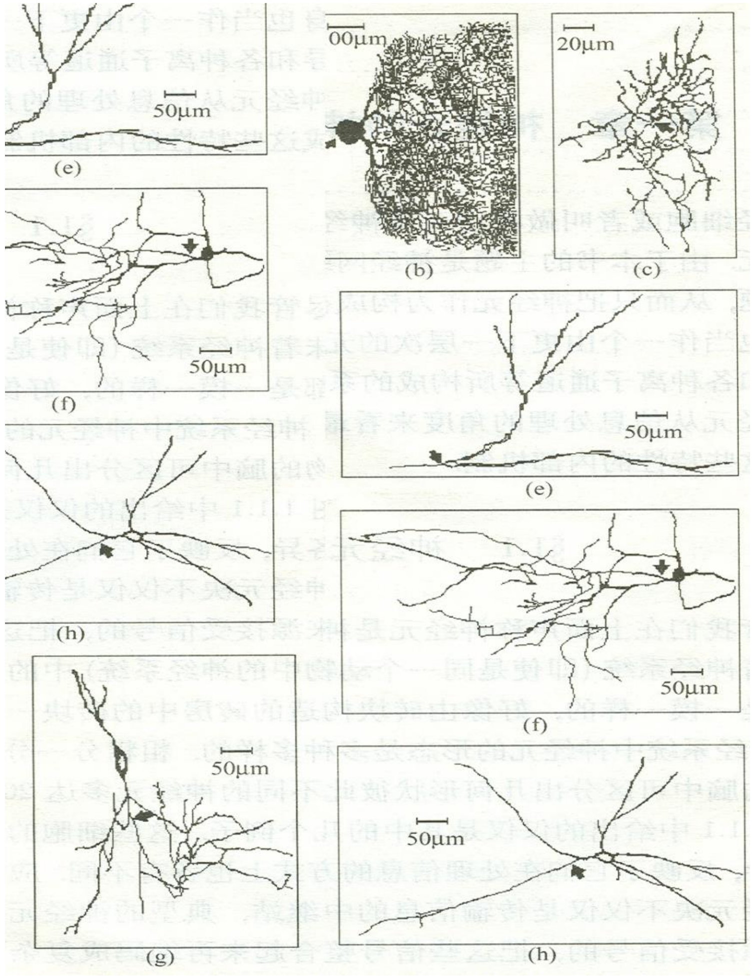
$$\hat{\tau}_{mixing} = \frac{m_{em}}{\dot{m}_{ex}} \frac{1}{4}$$

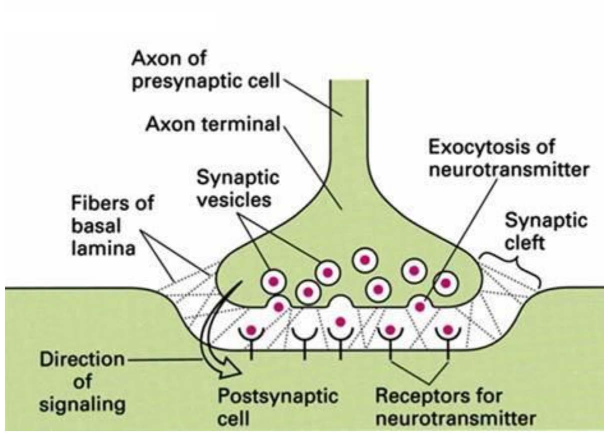
m_{em}: 排气管中的废气量
$\dot{m}_{ex}$: 废气的质量流率

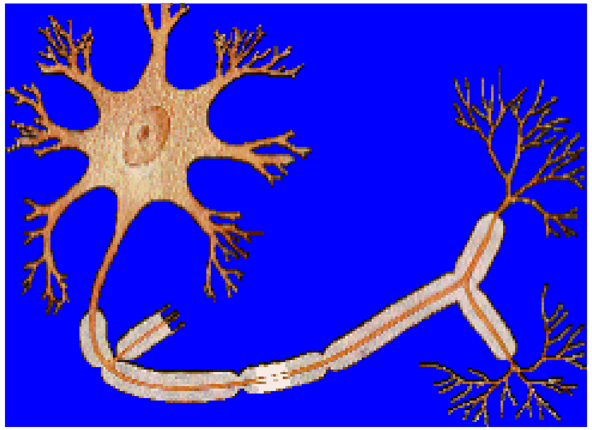
- **Binary sensor: Whether the excess air coefficient λ is 1**
- **Control goal: Excess air coefficient λ is 1, or AFR is 14.7**

- **How to realize more accurate AFR control with switching oxygen sensor (less used in industry now)**
- **How to use the rear oxygen sensor (mostly switch type) to diagnose the front sensor and catalyst**
- **How to realize good AFR control with tilt back sensor in fault mode**

Quantized systems: binary-valued data







- **McCulloch-Pitts model**

$$y_i = S\left(\sum_{j=1}^{n_{i1}} w_{ij}x_{ij} + \sum_{j=1}^{n_{i2}} \tilde{w}_{ij}\tilde{x}_{ij} - C_i\right)$$

- **Caianiello model**

$$x_i(t+1) = S\left(\sum_{j=1}^{N}\sum_{r=0}^{t} w_{ij}^{r} x_j(t-r) - C_i\right)$$

- **Hopfield model**
- **Nagumo-Sato model**
- **Aihara model**
- **……**

The i-th neuron's M-P model, the basis of many NN works

Excitation / Inhibition

Quantized systems: quantized data

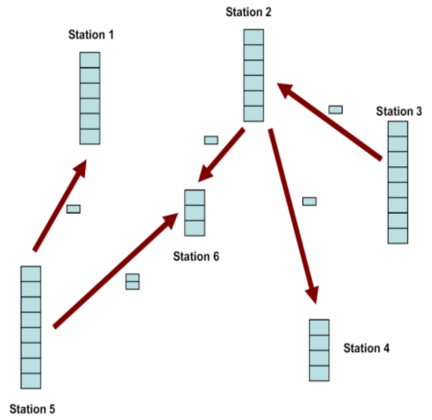
■ Wireless sensor networks(WSNs)

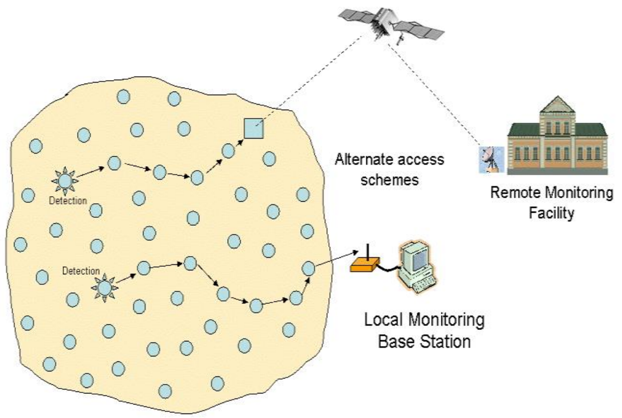
- ➢ Lower quality sensor
- ➢ Limited energy
- ➢ Limited bandwidth
- ➢ …..

Low cost and low power consumption WSNs is of great importance in military surveillance, environmental monitoring, healthy care, home & other commercial applications (Akyildiz et al, 2002)

Energy to transmit 1 bit ≈ **Energy for 1000 -- 3000 operations**

(Shnayder et al, 2004)





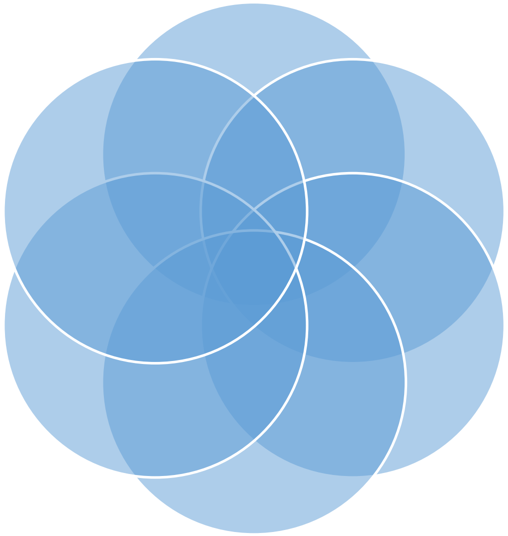
- ● Energy efficient algorithm for network coverage (Cardei & Wu, 2004; Krasnopeev et al, 2005)
- ● Distr. estim. & KF (Wong & Brockett, 1997; Reibeiro & Giannakis, 2006; K.Y. You et al. 2008)
- ● Consensus of networked systems (Aysal et al, 2008; Carli et al, 2010; Li et al, 2011; Meng et al, 2016)
- ● Decentralized detection (Xiao & Luo)

Classification of quantized systems

- Binary quantization: One-threshold
- Multi-layer quantization: Multi-threshold quantization

- Logarithmic quantization

- Fixed threshold quantization
- Time-varying threshold quantization



- Scalar quantization
- Vector quantization

- Uniform quantization
- Non-uniform quantization

- Finite threshold quantization
- Infinite threshold quantization

- Accurate data are hard to get, only coarse data or quantized data are available;
- Not economic to use accurate sensor/data, no need to use accurate data;
- Due to bandwidth limit, only quantized data can be got； ……

Feature of quantized systems

- **Binary-valued sensor**

$$q_k = \begin{cases} 1, \text{ if } y_k > C; \\ 0, \text{ if } y_k \le C. \end{cases}$$

$y_k \ge \mathrm{C}$ or $y_k < \mathrm{C}$

- **Set-valued sensors**

$$s = S(y) = \begin{cases} \alpha_1, \text{ if } y > C_1; \\ \alpha_j, \text{ if } C_j < y \le C_{j-1}, \\ \qquad j = 2,3,\ldots,m-1; \\ \alpha_m, \text{ if } y \le C_{m-1}. \end{cases}$$

- **Infinite threshold quantization**

$$s_k = \begin{cases} \vdots, \\ -\varepsilon, & y_n \in [-1.5\varepsilon, -0.5\varepsilon), \\ 0, & y_n \in [-0.5\varepsilon, 0.5\varepsilon), \\ \varepsilon, & y_n \in [0.5\varepsilon, 1.5\varepsilon), \\ \vdots, \end{cases}$$

Wide practical background:

- ✓ Sensor networks
- ✓ Integrated circuits
- ✓ Network communication
- ✓ Mechanical systems
- ✓ Smart material
- ✓ Automotive
- ✓ Chemical engineering
- ✓ Biology systems, …

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- ✓ **Sensor networks**
- ✓ **Integrated circuits**
- ✓ **Network communication**
- ✓ **Mechanical systems**
- ✓ **Smart material**
- ✓ **Automotive**
- ✓ **Chemical engineering**
- ✓ **Biology systems, …**

◆ **Wide Features: Less information, high nonlinearity**

- ✓ **Only the relationship of the concerned signal and the threshold can be obtained, not the value of the signal.**
- ✓ **Different from sampling, for sampled data, the data is accuracy.**
- ✓ **Different from the existing works based on quantization filtering and estimation, where some closed-loop conditions are required on quantization error, which depend on control and the performance of the closed-loop systems.**

Why do we consider quantized systems

- ✓ Need of practical systems
- ✓ Need of the development of control theory

Get a desired modelling and control goal with as less data as possible

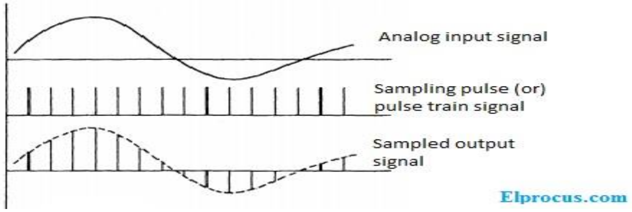
● Nyquist-Shannon sampling theorem

When converting from an analog signal to digital, the sampling frequency must be *greater than twice* the highest frequency of the input signal, in order to be able to reconstruct the original input signal perfectly from the sampled version.

(http://www.fact-index.com/n/ny/nyquist_shannon_sampling_theorem.html)

Claude Elwood Shannon
1916 – 2001
http://en.wikipedia.org/wiki/Claude_Shannon

Henry Nyquist
1889 - 1976
http://en.wikipedia.org/wiki/Harry_Nyquist



It tells us: in order to perfectly reconstruct an analog signal from its sampled version, how many sampled data are really needed.

- **Continuous processes:** T. Kailath, 1980; L. Arnold, 1974; J.C. Doyle *et al*., 2013; ……
- **Sampling data:** Nyquist-Shannon sampling theorem, periodic/non-periodic sampling, ……
- **Quantized data:** R.E. Curry, 1970; A. Gersho & R.M. Gray, 1991; L.Y. Wang et al., 2003; M.Y. Fu et al., 2009; Godoy et al., 2011; K.Y. You, 2015; Z.P. Jiang & T. Liu, 2018; ……
- **Event-driven systems:** K. Johansson et al., 2012; G. Feng, 2013; Z.P. Jiang, 2015; ……

How to use less data to reach a desired modelling and control goal ?

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State-feedback control

Plant **Control**

$$\dot{x} = Ax + Bu \quad \Rightarrow \quad u = Kx$$

Pole-placement,
Stabilization,
LQ control,
Robust control, etc.

Key assumption： States are known!

State-feedback control

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$$\dot{x} = Ax + Bu \Rightarrow u = Kx$$

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Output-feedback control

Plant | **Control**

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du; \end{aligned} \Rightarrow u = Ky;$$

$$\begin{cases} \dot{\hat{x}} = Ny + Fu, \\ u = K\hat{x} + Hy; \end{cases}$$

Static output-feedback,
Dynamic output-feedback,
……

Key assumption： Outputs are known!

Filtering, identification, adaptive control

- **System model:**

  * $\dot{x} = Ax + Bu$ —— $\hat{x} = x + w$ —— Measurement noises

  * $y_k = \sum_{i=1}^{n} a_i y_{k-i} + \sum_{i=1}^{n} b_i u_{k-i} + d_k = \theta^\tau \varphi_k + d_k$

- **Estimation method: LS, SG, LMS, KF, ……**

  When y_k is known, we can use LS to estimate the parameter :

  $$\hat{\theta}_k = \arg\min_{\theta \in R} \sum_{i=1}^{k} (y_i - \theta^\tau \varphi_{i-1})^2;$$

  and use the certainty equivalence principle to design control.

- **Key assumption: States/outputs are known!**

Basic problems of quantized systems

■ **How to realize satisfactory identification and control with quantized data?**

➢System identification

- **Identifiability**
- **Identification methods**
- **Estimation error, conv. rate**
- **Input/quantizer design**
- **Uncertainty influence on est. error, conv. rate, computing complexity**
- **(Asymptotic) Efficiency**
- **……**

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State estimation

- State estimation
- Convergence performance
- Computation complexity
- Threshold’s influence on est. error, conv. rate, computing complexity
- ……

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Control synthesis

- Stabilization
- Robust control
- Output-feedback
- Adaptive control
- Consensus control
- ……

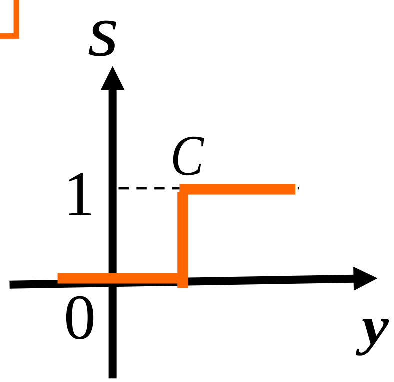
Identifiability

- **Binary set-valued sensor**

Threshold

$$y_k \geq C \text{ or } y_k < C$$

$$q_k = \begin{cases} 1, \text{ if } y_k > C; \\ 0, \text{ if } y_k \leq C. \end{cases}$$



When y_k is known： $y_k = b\,u_k \Rightarrow b = y_1 / u_1$

When y_k is unknown： $y_k = b\,u_k \not\Rightarrow b = y_1 / u_1$

Identifiability

■ Guess a word

- Given a book of 400 pages, with no more than 1000 words on each page. (Thus, totally there is no more than 0.4 million words in the book.)
- You can choose a word from the book randomly, remember it by yourself and do not tell me which it is.
- I can get the word that you choose by asking you no more than 20 questions, and you can answer each question only by “Yes” or “No”.
- Using binary observations to get good (even exact) estimation

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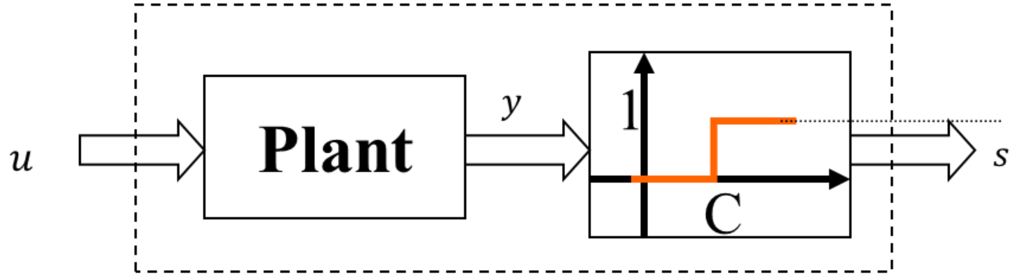
Parameter identification with quantized data

- **Bisection method and parameter decoupling**
- **Likelihood method**
- **Expectation maximization method**
- **Empirical measure method with/without truncation**
- **Recursive algorithms:**

  Stochastic approximation, sign-error, CRLB, Quasi-Newton based

Parameter identification with binary-valued data

■ Model:



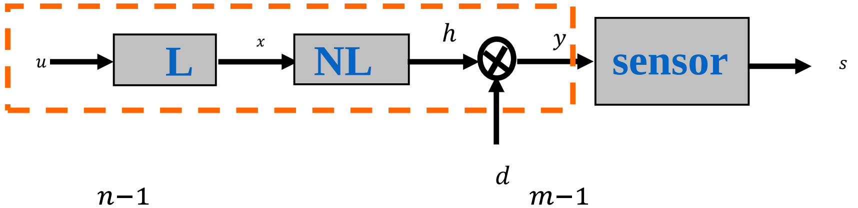
$$y(k) = P(y, u, \theta) + d(k),$$

$$s(k) = \begin{cases} 0, y(k) > C; \\ 1, y(k) \le C. \end{cases}$$

● **ARMAX:** $y(t) = \sum_{i=1}^{n} a_i y(t-i) + \sum_{i=0}^{n} b_i u(t-i) + d(t)$

a_i, b_i–unknown parameters, $d(t)$ – noises

● **Wiener system:**



$$x(t) = \sum_{i=0}^{n-1} a_i u(t-i), \qquad y(t) = \sum_{i=0}^{m-1} x^i(t) b_i + d(t)$$

● **Hammerstein system:**

$$\begin{cases} y(t) = \sum_{i=0}^{n-1} a_i x(t-i) + d(t), \\ x(k) = b_0 + \sum_{j=0}^{m-1} b_j u^j(t), b_m = 1. \end{cases}$$

■ Goal:

Estimate the unknown parameters by using binary-valued data.

Identification with binary-valued data

- **Model:** $y(t) = bu(t)$

$b \in [\underline{b}(0), \overline{b}(0)],$

$0 < \underline{b}(0) < \overline{b}(0).$

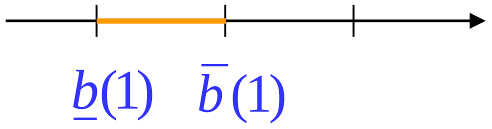
Initial value

- **Input:** $u(1) = \dfrac{2C}{\underline{b}(0) + \overline{b}(0)}$

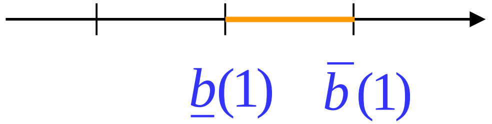
Binary

- **1-step estimate:**

If $y(1) \le C$, then $b \le \overline{b}(1) = \dfrac{\underline{b}(0) + \overline{b}(0)}{2}$;

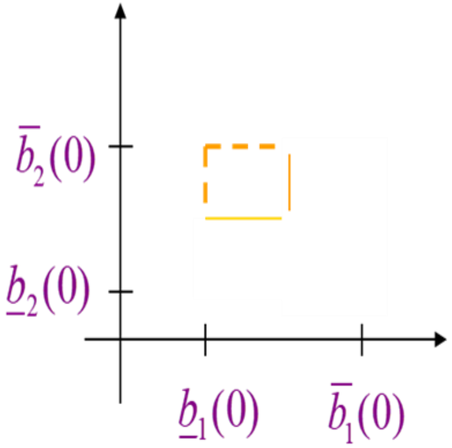


If $y(1) > C$, then $b > \underline{b}(1) = \dfrac{\underline{b}(0) + \overline{b}(0)}{2}$.



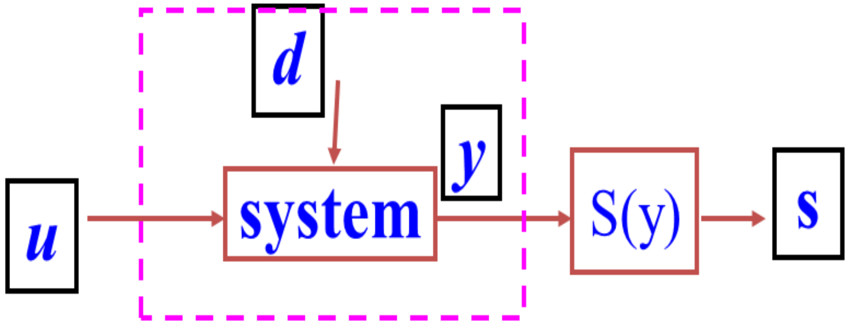
- **Model:** $y(t) = b_1 u(t) + b_2 u(t-1),$

$$y(1) = b_1 u(1),$$
$$y(2) = b_1 u(2) + b_2 u(1),$$
$$y(3) = b_1 u(3) + b_2 u(2),$$



$u(3) = 0$
$u(0) = 0$

$$y(1) = b_1 u(1),$$
$$y(3) = b_2 u(2).$$



$$y(t) = \sum_{i=1}^{n} b_i u(t-i) + d(t)$$

$$s = S(x) = \begin{cases} 1, & \text{if } x > C_1; \\ j, & \text{if } C_j < x \le C_{j-1}, \\ & \quad j = 2,3,\ldots,m-1; \\ m, & \text{if } x \le C_{m-1}. \end{cases}$$

Deterministic framework

- **Model:** $y(t) = \sum_{i=1}^{n} b_i u(t-i) + d(t)$

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- **Theorem:** For FIR system, suppose $|d(t)| \le \delta$,

Let $\overline{b}(0) = \max_{1\le i\le n} \overline{b}_i(0)$, $\underline{b}(0) = \min_{1\le i\le n} \underline{b}_i(0)$, $\varepsilon(0) = \mathrm{Rad}(\Omega_0)$,

$$\sigma = \frac{m(C_1 - C_2) + 2\delta}{2C_1 - C_2 - \delta}\overline{b}(0),\ \ \alpha_1 = \frac{(C_1 - C_{m-1} - 2\delta)\underline{b}(0)}{C_1 - \delta},$$

$\alpha_2 = (C_{m-1} + \delta)(C_1 + C_{m-1})^{-1}$, $C_1 - C_2 = \dots = C_{m-2} - C_{m-1}$.

Then $\forall \varepsilon \in (\sigma, \varepsilon(0))$, we have

$$N(\varepsilon) \le \frac{\ell_n \ln \dfrac{\alpha_1 + \varepsilon n^{-1/2}}{(1-\alpha_1)\varepsilon(0)}}{\ln \alpha_2}.$$

- **Method: Bisection + Parameter decoupling**

Deterministic framework

- **Model:** $y(t) = \sum_{i=1}^{n} b_i u(t-i) + d(t)$

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- **Method: Bisection + Parameter decoupling**

- **Model:** $\boldsymbol{y(t) = \theta u(t) + d(t)}$

- **Bisection method**

$$u_k = \frac{2C}{\underline{\theta}_{k-1} + \bar{\theta}_{k-1}},$$

$$\begin{cases} \bar{\theta}_k = \bar{\theta}_{k-1}, \underline{\theta}_k = \dfrac{C-\varepsilon}{u_k}, if\ s_k = 0, \\ \bar{\theta}_k = \dfrac{C+\varepsilon}{u_k}, \underline{\theta}_k = \underline{\theta}_{k-1}, if\ s_k = 1, \end{cases}$$

where $\theta \in [\underline{\theta}_0, \bar{\theta}_0]$, $|d_k| \le \varepsilon \le C$, $n = 1$

- **Algorithm properties**
  - the irreducible relative error
  - the time complexity
  - exponential convergent for noise-free case

* L. Y. Wang, J. F. Zhang & G. Yin, IEEE TAC, 2003

Follow-on work: *M. Casini, A. Garulli & A. Vicino, CDC, 2007;
*M. Casini, A. Garulli and A. Vicino, IEEE TAC, 2011

Stochastic framework: Likelihood method

■ Normal LS method

$$y_k = \sum_{i=1}^{n} a_i y_{k-i} + \sum_{i=1}^{n} b_i u_{k-i} + d_k = \theta^{\tau} \varphi_k + d_k$$

When y is know, the LS is:

$$\hat{\theta}_k = \arg\min_{\theta \in R} \sum_{i=1}^{k} (y_i - \theta^{\tau} \varphi_{i-1})^2;$$

and use the certainty equivalence principle to design control.

$$y(t) = \sum_{i=0}^{n} b_i u(t-i) + d(t)$$

■ Likelihood function

$$\begin{aligned}
\theta &= \operatorname{argmax}_{\theta \in \Omega} Pr\left(s_{1:N} | \phi_{1:N}, \theta\right) \\
&= \operatorname{argmax}_{\theta \in \Omega} \log Pr\left(s_{1:N} | \phi_{1:N}, \theta\right) \\
&= \operatorname{argmax}_{\theta \in \Omega} \sum_{k=1}^{N} \log Pr\left(s_k | \phi_k, \theta\right)
\end{aligned}$$

The solution is

$$\sum_{k=1}^{N} \left(\frac{1}{F_k} I_{\{s_k=1\}} - \frac{1}{1-F_k} I_{\{s_k=0\}} \right) f_k \phi_k^T = 0$$

where

$$F_k = F(C - \phi_k^T \theta) \quad f_k = f(C - \phi_k^T \theta)$$

● **Difficulty: there is no explicit solution!**

- **Likelihood function:**

$$\begin{aligned}\theta &= \mathrm{argmax}_{\theta\in\Omega} Pr\left(s_{1:N}|\phi_{1:N},\theta\right)\\ &= \mathrm{argmax}_{\theta\in\Omega} \log Pr\left(s_{1:N}|\phi_{1:N},\theta\right)\\ &= \mathrm{argmax}_{\theta\in\Omega} \sum_{k=1}^{N} \log Pr\left(s_k|\phi_k,\theta\right)\end{aligned}$$

- **The solution:**

$$\sum_{k=1}^{N}\left(\frac{1}{F_k}I_{\{s_k=1\}} - \frac{1}{1-F_k}I_{\{s_k=0\}}\right) f_k\phi_k^T = 0$$

$$F_k = F(C-\phi_k^T\theta) \quad f_k = f(C-\phi_k^T\theta)$$

- **Difficulty: no explicit solution!**

- **Cramér-Rao lower bound:**

$$\Delta_k = \left(\sum_{i=1}^{k}\lambda_i\phi_i\phi_i^T\right)^{-1} \mathit{where}\ \lambda_i = \frac{f_i^2}{F_i(1-F_i)}$$

- **The ideal algorithm (CRLB based):**

$$\begin{aligned}\hat{\theta}_k &= \hat{\theta}_{k-1} - P_{k-1}\phi_k\tilde{s}_k\\ \tilde{s}_k &= \hat{\lambda}_k\left(s_k - F\left(C-\phi_k^T\hat{\theta}_{k-1}\right)\right)\\ P_k &= P_{k-1} - \hat{f}_k\hat{\lambda}_k P_{k-1}\phi_k\phi_k^T P_{k-1}\end{aligned}$$

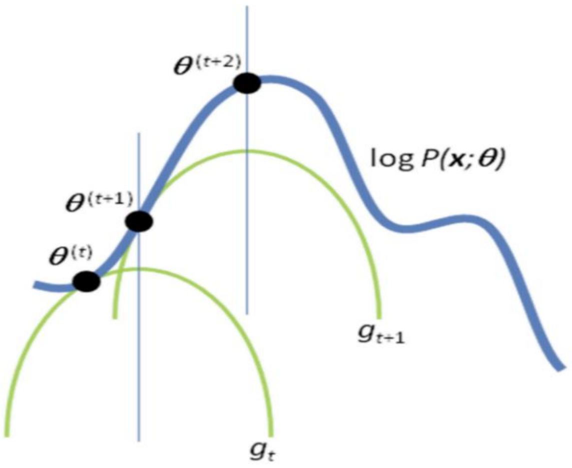
$$\hat{f}_k = f(C-\phi_k^T\hat{\theta}_{k-1})$$

$$\hat{\lambda}_k = \frac{f(C-\phi_k^T\hat{\theta}_{k-1})}{F(C-\phi_k^T\hat{\theta}_{k-1})(1-F(C-\phi_k^T\hat{\theta}_{k-1}))}$$

Expectation maximization method

- **Likelihood function:**

$$\begin{aligned}\theta &= \operatorname{argmax}_{\theta\in\Omega} Pr\left(s_{1:N}|\phi_{1:N},\theta\right)\\ &= \operatorname{argmax}_{\theta\in\Omega} \log Pr\left(s_{1:N}|\phi_{1:N},\theta\right)\\ &= \operatorname{argmax}_{\theta\in\Omega} \sum_{k=1}^{N} \log Pr\left(s_k|\phi_k,\theta\right)\end{aligned}$$



Supplementary Figure 1 Convergence of the EM algorithm. Starting from initial parameters $\theta^{(t)}$, the E-step of the EM algorithm constructs a function g_t that lower-bounds the objective function $\log P(x;\theta)$. In the M-step, $\theta^{(t+1)}$ is computed as the maximum of g_t. In the next E-step, a new lower-bound g_{t+1} is constructed; maximization of g_{t+1} in the next M-step gives $\theta^{(t+2)}$, etc.

***D. Marelli, K.Y. You & M.Y. Fu, Automatica, 2013**

*** B. Godoy, G. Goodwin, J. Aguero, D. Marelli & T. Wigren, *Automatica*, 2011**

$$\theta = \left[\sum_{t=1}^{N}\varphi_t R^{-1}\varphi_t^T\right]^{-1}\sum_{t=1}^{N}\varphi_t R^{-1}\hat{x}_t,$$

$$R = \frac{1}{N}\sum_{t=1}^{N}\left[R_i^{1/2}\frac{I_t^{(2)}}{I_t^{(0)}}R_i^{1/2} + 2\varphi_t^T(\hat{\theta}_i-\theta)\frac{I_t^{(1)}}{I_t^{(0)}}R_i^{1/2} + \varphi_t^T(\hat{\theta}_i-\theta)(\hat{\theta}_i-\theta)^T\varphi_t\right],$$

*** Y. L. Zhao, W. J. Bi & T. Wang, SCIS, 2016**

$$\begin{aligned}&\widehat{\theta}_N(t+1)\\ =&\widehat{\theta}_N(t) - \Big(\sum_{k=1}^{N}\phi_k\phi_k^T\Big)^{-1}\Big(\sum_{k=1}^{N}\phi_k\cdot f\big(C-\phi_k^T\widehat{\theta}_N(t)\big)\\ &\cdot\Big[\frac{I_{\{s_k=1\}}}{F\big(C-\phi_k^T\widehat{\theta}_N(t)\big)} - \frac{I_{\{s_k=0\}}}{1-F\big(C-\phi_k^T\widehat{\theta}_N(t)\big)}\Big]\Big),\end{aligned}$$

Empirical measure method with set-valued data

- **Empirical measure method:**

$$\xi_N = \frac{1}{N}\sum_{l=1}^{N} S_l = \frac{1}{N}\sum_{l=1}^{N} \boldsymbol{I}_{\{D_l \le C\vec{1} - \Phi\theta\}} \to \xi = \boldsymbol{F}(C\vec{1} - \Phi\theta),$$

$$\begin{pmatrix} v_n & v_{n-1} & \cdots & v_1 \\ v_1 & v_n & & v_2 \\ & \ddots & \ddots & \\ v_{n-1} & v_{n-2} & \cdots & v_n \end{pmatrix}$$

- **Assumption:** $C, F(\cdot)$ are known

$$\hat{\theta}_N = \Phi^{-1}\left[C\vec{1} - \boldsymbol{F}^{-1}(\xi_k)\right] \to \theta.$$

where $F(\cdot)$ is the PDF of noises, $\Phi_0 = [\varphi_1, \cdots, \varphi_n]^T$,
φ_k is the n-period input;

- **Algorithm properties:**
  - **Convergence:**

$$\theta(N) \to \theta \quad \text{w.p.1.}$$

  - **Convergence rate**

$$\sigma^2(N) = O(1/N).$$

  - **Efficiency:**

$$N[\sigma^2(N) - \sigma^2_{CR}(N)] \to 0.$$

* L. Y. Wang, J. F. Zhang & G. Yin, IEEE TAC, 2003
* Y. L. Zhao,, L. Y. Wang, G. Yin & J. F. Zhang, Automatica, 2010

● Empirical measure method (with truncation)

$$\xi_k^i = \frac{1}{k}\sum_{l=1}^{k-1} s_{ln+i},$$

$$L_k = \left[C - G(\xi_k^1), \cdots, C - G(\xi_k^n)\right]^T,$$

$$G(\xi_k^i) = \begin{cases} z, \xi_k^i < z, \\ F^{-1}(\xi_k^i), z \le \xi_k^i \le 1 - z, \\ z, \xi_k^i > 1 - z, \end{cases}$$

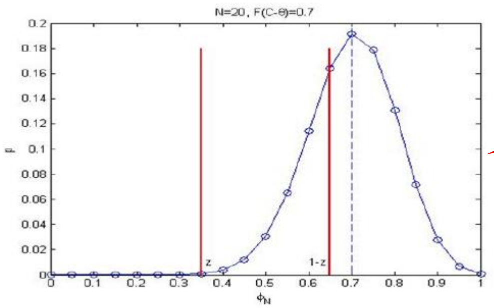
$$\hat{\theta}_k = \Phi^{-1} L_k.$$

$$\begin{pmatrix} v_n & v_{n-1} & \cdots & v_1 \\ v_1 & v_n & & v_2 \\ & \ddots & \ddots & \\ v_{n-1} & v_{n-2} & \cdots & v_n \end{pmatrix}$$

where z is chosen by

$$z < p_{i,j} = F(C_i - \zeta_j) < 1 - z$$

and ζ_j depends on the true parameter.



The true parameter range must be known

● Algorithm properties:

➢ **Convergence:**

$$\hat{\theta}_k \to \theta, \mathrm{w.p.}\,1$$

➢ **Convergence rate**

$$\lim_{N\to\infty} N(\sigma_j^{2*}(N) - \sigma_{CR,j}^2(N)) = 0$$

➢ **Main Idea of the proof:**

Taylor expansion;

Uniformly bounded of the probability density function ;

*** Y. L. Zhao, J. F. Zhang, L. Y. Wang & G. Yin, SIAM J. Control & Optim, 2010**

• Empirical measure method (without truncation)

$$\xi_k^i = \frac{1}{k}\sum_{l=1}^{k-1} s_{ln+i},$$

$$\zeta_k^i = \begin{cases} 1/2, \xi_k^i = 0, \\ \xi_k^i, 0 < \xi_k^i < 1, \\ 1/2, \xi_k^i = 1, \end{cases}$$

$$L_k = \left[C - F^{-1}(\zeta_k^i), \cdots, C - F^{-1}(\zeta_k^i)\right]^T,$$

$$\hat{\theta}_k = \Phi_0^{-1} L_k.$$

$$\begin{pmatrix} v_n & v_{n-1} & \cdots & v_1 \\ v_1 & v_n & & v_2 \\ & \ddots & \ddots & \\ v_{n-1} & v_{n-2} & \cdots & v_n \end{pmatrix}$$

• Algorithm properties:

- **Mean square convergence rate**

$$E(\hat{\theta}_N - \theta)^2 = O\left(\frac{1}{N}\right)$$

The convergence rate is at the same order as that of accurate measurements

- **Main idea of the proof:**

$$\sum_{\frac{1}{N} \le \frac{l}{N} < \epsilon} F^{-1}(l/N)\, C_N^l \delta^l (1-\delta)^{N-l} = O(e^{-d_1 N}),$$

$$\sum_{1-\epsilon < \frac{l}{N} \le \frac{N-1}{N}} F^{-1}(l/N)\, C_N^l \delta^l (1-\delta)^{N-l} = O(e^{-d_2 N}).$$

*** Y. Zhao, T. Wang & W. Bi, IEEE TAC, 2019**

Recursive projection algorithm

● Stochastic approximation type:

$$\begin{cases} \hat{\theta}_{k+1} = \Pi_{\Theta}\left(\hat{\theta}_k + \dfrac{\beta\varphi_k}{r_{k+1}}\left(F\left(C - \varphi_k^T\hat{\theta}_k\right) - s_{k+1}\right)\right), \\ r_{k+1} = 1 + \sum_{i=1}^{k} \varphi_i^T\varphi_i \,. \end{cases}$$

where $\Pi_{\Theta}(\cdot)$is the projection from $\mathbb{R}^n$ to Θ, defined as

$$\Pi_{\Theta}(\xi) = \arg\min_{\zeta\in\Theta}\|\xi - \zeta\|, \forall\xi \in \mathbb{R}^n$$

- **Assumption**: The inputs $\{\varphi_k, k = 1,2,\cdots\}$ satisfy $\sup_{k\geq 1}\|\varphi_k\| \triangleq M < \infty$. Besides,

$$\frac{1}{N}\sum_{i=k}^{k+N-1} \varphi_i\varphi_i^T \geq \delta^2 I.$$

*** J. Guo & Y. L. Zhao, Automatica, 2013**

*** T. Wang, M. Hu & Y. L. Zhao, CAC, 2018**

- **Difficulty:**

$$E\tilde{\theta}_k^T\tilde{\theta}_k = E\sum_{l=1}^{k}\phi_l P_{l-1}W_{l:k}^T W_{l:k}P_{l-1}\alpha_l^2\phi_l\left(F_l - s_l\right)^2 + \tilde{\theta}_0^T W_{1:k}^T W_{1:k}\tilde{\theta}_0$$
$$+2E\sum_{l=1}^{k}\tilde{\theta}_{l-1}^T W_{l:k}^T W_{l:k}P_{l-1}\alpha_l\phi_l\left(F_l - s_l\right)$$
$$= O(1/k)$$

Cross item

- **Convergence:** mean-square and almost surely convergent, i.e.,

$$\lim_{k\to\infty}\mathrm{E}(\hat{\theta}_k - \theta)^T(\hat{\theta}_k - \theta) = 0$$

$$\lim_{k\to\infty}\hat{\theta}_k = \theta, \qquad \text{a.s.}$$

- **Convergence Rate:**

$$\mathrm{E}(\hat{\theta}_k - \theta)^T(\hat{\theta}_k - \theta) = O(1/k)$$

Recursive projection algorithm

- **Sign-error based:**

The threshold of binary quantizer is design as time-varying threshold $\varphi_k^T\hat{\theta}_k$, i.e.,

$$s_{k+1}=I_{\{y_{k+1}>\varphi_k^T\hat{\theta}_k\}}-I_{\{y_{k+1}<\varphi_k^T\hat{\theta}_k\}}$$

Then the algorithm is

$$\begin{cases}\hat{\theta}_{k+1}=\Pi_\Theta\left(\hat{\theta}_k+\dfrac{\beta\varphi_k}{r_{k+1}}s_{k+1}\right)\\ r_{k+1}=\sum_{i=1}^k\varphi_i^T\varphi_i\end{cases}$$

where $\Pi_\Theta\ (\cdot)$is the projection from $\mathbb{R}^n$ to Θ, defined as

$$\Pi_\Theta\ (\xi)=\arg\min_{\zeta\in\Theta}\|\xi-\zeta\|,\forall\xi\in\mathbb{R}^n$$

- **Assumption**: The inputs $\{\varphi_k,k=1,2,\cdots\}$ satisfy $\sup_{k\geq1}\|\varphi_k\|\triangleq M<\infty$, and

$$\frac{1}{N}\sum_{i=k}^{k+N-1}\varphi_i\varphi_i^T\geq\delta^2I.$$

- **Properties:**

➢ For noise-free case with PE condition, square convergence rate is $O\left(\frac{1}{k^2}\right)$;

➢ For bounded noise case, an upper bound of the estimation error is given in terms of the noise bound and the lower bound of the PE condition.

➢ For stochastic noise case, mean-square and almost surely convergence are obtained, mean square conv. rate is $O\left(\frac{1}{k}\right)$.

* Y. Wang, Y. Zhao, J. F. Zhang & J. Guo, submitted to Automatica, 2021

Recursive projection algorithm

CRLB based:

$$\begin{cases} \hat{\theta}_{k+1} = \Pi_\Theta\left(\hat{\theta}_k + \alpha_k P_k \varphi_k \left(F\left(C - \varphi_k^T \hat{\theta}_k\right) - s_{k+1}\right)\right) \\ P_{k+1} = P_k - \dfrac{\beta_k P_k \varphi_k \varphi_k^T P_k}{1 + \beta_k \varphi_k^T P_k \varphi_k} \end{cases}$$

where $\alpha_k = \frac{\hat{f}_k}{\hat{F}_k(1-\hat{F}_k)}$, $\beta_k = \frac{\hat{f}_k^2}{\hat{F}_k(1-\hat{F}_k)}$,

$$\hat{f}_k = f\left(C - \varphi_k^T \hat{\theta}_k\right), \hat{F}_k = F\left(C - \varphi_k^T \hat{\theta}_k\right)$$

- **Assumption**: The inputs $\{\varphi_k, k = 1,2,\cdots\}$ satisfy $\sup_{k\geq 1}\|\varphi_k\| \triangleq M < \infty$. Besides,

$$\liminf_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \varphi_i \varphi_i^T > 0.$$

- **Convergence:**

For 1-order system with binary-valued observations, the algorithm is mean square convergent, i.e.

$$\lim_{k\to\infty} E\tilde{\theta}_k^2 = 0.$$

- **Convergence Rate:**

the mean square convergence rate is $O\left(\frac{1}{k}\right)$

- **Asymptotically efficient:**

$$\lim_{k\to\infty} \Delta_k^{-1}(E\tilde{\theta}_k^2 - \Delta_k) = 0.$$

where

$$\Delta_k = \left(\sum_{i=1}^{k} \rho_i \phi_i \phi_i^T\right)^{-1}$$

is the CR lower bound with $\rho_i = \dfrac{f_i^2}{F_i(1 - F_i)}$

- **Difficulty of high-order: Compression matrix -- random, correlated, with unknown parameters**

$$W_{l:k} = \prod_{j=l+1}^{k} (I - \delta_j P_{j-1} \phi_j \phi_j^T) = \prod_{j=l+1}^{k} W_j$$

$$\delta_j = \alpha_j f(C - \phi_j^T \check{\theta}_{j-1})$$

*** H Zhang, T. Wang & Y. Zhao, IEEE SMC, 2019**

Recursive projection algorithm

● Quasi-Newton type:

$$\hat{\theta}_{k+1} = \Pi_{P_{k+1}^{-1}}\left\{\hat{\theta}_k + a_k\beta_k P_k\phi_k e_{k+1}\right\},$$

$$P_{k+1} = P_k - \beta_k^2 a_k P_k\phi_k\phi_k^\tau P_k,$$
$$e_{k+1} = s_{k+1} - 1 + F_{k+1}(c_k - \phi_k^\tau\hat{\theta}_k),$$
$$a_k = \frac{1}{1+\beta_k^2\phi_k^\tau P_k\phi_k},$$
$$0 < \beta_{k+1} \le \min\left\{\beta_k, \inf_{|x|\le LM+C} f_{k+2}(x)\right\},$$

where $\Pi_Q\ (\cdot)$is the projection from $\mathbb{R}^n$ to Θ given by

$$\Pi_\Theta\ (\xi) = \arg\min_{\zeta\in\Theta}\|\xi-\zeta\|_Q\,, \forall\xi\in\mathbb{R}^n,$$

and $\|x\|_Q = \sqrt{x^T Q x}$ for $x \in \mathbb{R}^n$.

● **Weak excitation condition**: The input sequence $\{\phi_k, \mathcal{F}_k\}$ satisfies $\sup_{k\ge1}\|\phi_k\| \triangleq M < \infty$, a. s., and

$$\left\{\log\lambda_{\max}\left(\sum_{i=1}^n \phi_i\phi_i^T\right)\right\}/\lambda_{\min}\left(\sum_1^n \phi_i\phi_i^T\right) \to 0. \quad a.s.$$

● **Convergence:**

The estimate is convergent under non-PE condition

$$\left\|\tilde{\theta}_{n+1}\right\|^2 = O\left(\frac{\log\left(\lambda_{\max}\left\{P_{n+1}^{-1}\right\}\right)}{\lambda_{\min}\left\{P_{n+1}^{-1}\right\}}\right), \quad \text{a.s.}$$

*** L.T. Zhang, Y. L. Zhao & L. Guo, submitted to Automatica, 2021**

*** D. Marelli, K.Y. You & M.Y. Fu, Automatica, 2013**

The scalar gains⇒ The matrix gains; PE condition ⇒ Weak excitation condition

Parameter identification with quantized data

- Bisection method and parameter decoupling for noise-free or bounded noises
- Likelihood method for the case with stochastic noises
- Expectation maximization method
- Empirical measure method with/without truncation
- Recursive projection algorithm:
  - ✓ Stochastic approximation: scalar step, known distr. function
  - ✓ Sign-error: scalar step, time-varying threshold, unknown distr. function
  - ✓ CRLB based: matrix step depending on estimate, asymptotic efficiency
  - ✓ Quasi-Newton: matrix step, weak excitation condition

Applications of quantization identification

- **Complex disease modeling (Peking University Sixth Hospital, St. Judy Children's Research Hospital, USA, etc.)**
  - Establish a set-valued model of schizophrenia and leukemia
  - Construct a more effective statistical verification method

- **Radar target recognition (Academies of Astronautics)**
  - Establish a set-valued model of radar target recognition
  - Construct an intelligent recognition algorithm based on quantized estimation

- **Satellite control (Beijing Institute of Control Engineering)**
  - Construct an estimation algorithm under saturation constraints
  - Realize the joint control of the auto-disturbance position and attitude of the towed satellite

Applications of quantization identification

■ Association analysis of gene based on quantized estimation [SMMR 2019 et al]

- Acute lymphocytic leukemia (St. Judy Children's Research Hospital)
- 2024 cases of European descent: statistical p-value 0.000996
- Found a new site rs2893881 in ARID5B gene

Table 4. Single nucleotide polymorphisms (SNPs) associated with ALL susceptibility in White and Hispanic

	White			Hispanic		
SNP	**MA**	**LG**	**SV**	**MA**	**LG**	**SV**
rs10821936	C	8.34×10^{-20}	2.70×10^{-20}	T	1.03×10^{-7}	6.99×10^{-8}
rs10821938	A	1.47×10^{-14}	8.89×10^{-15}	C	4.27×10^{-7}	2.74×10^{-7}
rs10994982	G	2.66×10^{-7}	2.41×10^{-7}	G	3.81×10^{-6}	3.47×10^{-6}
rs7087125	T	9.22×10^{-6}	8.76×10^{-6}			
rs7896246	A	1.03×10^{-19}	3.32×10^{-20}	G	2.77×10^{-7}	2.24×10^{-7}
rs7923074	A	1.50×10^{-13}	9.85×10^{-14}	C	2.09×10^{-7}	1.35×10^{-7}
rs2893881	G		0.000996			

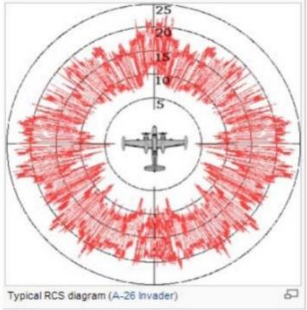
MA: minor allele.

Construct a new and more effective statistical verification method

More reasonable assumption and more reliable for small sample size

Applications of quantization identification

Radar target recognition based on quantized identification method



- The target type is "true, false", generating quantized data
- Establish a model of the characteristics of radar data and the authenticity of the target
- Obtain the main radar characteristics and recognition rate

Covariance of noise	Experiment number	Fuzzy Classification	Evidential Reasoning	Quantized identification
0.05	1	0.96	0.93	0.99
	2	0.91	0.94	0.99
0.5	1	0.79	0.89	0.97
	2	0.73	0.91	0.98
1	1	0.71	0.79	0.93
	2	0.70	0.85	0.94

Compared with other methods, the algorithm is still reliable at low signal-to-noise ratio

The required data size is small; The results can be explained.

Kalman filter with quantized data

Stochastic approximation with sign-error

Sign-error algorithm (binary reinforcement (BR) /sign algorithm):

Cost function:

$$L(H) = \mathbb{E}|y_k - H^T X_k|^2$$

Recursive algorithm (SA type):

$$H_{k+1} = H_k + a_k X_k\left(y_k - H_k^T X_k\right)$$

where $\sum_k a_k = \infty, a_k \to 0$ as $k \to 0$.

Cost function:

$$L(H) = \mathbb{E}|y_k - H^T X_k|$$

$$L_H(H) = -\mathbb{E}(X_k \text{sign}(y_k - H^T X_k))$$

Recursive sign algorithm

$$H_{k+1} = H_k + a_k X_k \text{sign}(y_k - H_k^T X_k)$$

1964 — M. Aizerman & E. Braverman & L. Rozonoer: Found BR algorithm

1972 — A. Gersho, IEEE fellow

1984 — A. Gersho, IEEE fellow: Convergent under i.i.d. signals

1989 — E. Eweda, IEEE fellow

1991 — E. Eweda, IEEE fellow: Convergent under M-dependent signals

2003 — H. F. Chen & G. Yin, IEEE fellow: Convergent under stationary ergodicity; sign-error algorithms with expanding truncation bounds

2012 — B. Csáji & E. Weyer

2015 — K. You

2017 — W. Zhao & H. F. Chen & R. Tempo & F.Dabbene,

Applied to binary measurements with i.i.d. inputs

Kalman filter with quantized innovation

● **Model:** $\mathbf{x}(n) = \mathbf{A}(n)\mathbf{x}(n-1) + \mathbf{w}(n)$

$$y(n) = \mathbf{h}^T(n)\mathbf{x}(n) + v(n)$$

● **State estimation:**

$$\hat{\mathbf{x}}(n|n) := E[\mathbf{x}(n)|\mathbf{b}_{0:n}] = \int_{\mathbb{R}^p} \mathbf{x}(n)p[\mathbf{x}(n)|\mathbf{b}_{0:n}]d\mathbf{x}(n)$$

● **Results:**

* General multi-level quantized innovation KF
* Optimal MLQ-KF w.r.t. quantization levels
* Optimal filter is in terms of Riccati difference eq.
* Convergence of the MLQ-KF is established.
* For 1-bit trans. case, better performance than the sign of innovation filter given (Ribeiro, 2006)

● **Quantizer:**

$$b(n) := \begin{cases} z_N, & \bar{z}_N < \epsilon(n) \\ z_{N-1}, & \bar{z}_{N-1} < \epsilon(n) \leq \bar{z}_N \\ \vdots & \vdots \\ z_1, & \bar{z}_1 < \epsilon(n) \leq \bar{z}_2 \\ 0, & -\bar{z}_1 < \epsilon(n) \leq \bar{z}_1 \\ -z_1, & -\bar{z}_2 < \epsilon(n) \leq -\bar{z}_1 \\ \vdots & \vdots \\ -z_N, & \epsilon(n) \leq -\bar{z}_N \end{cases}$$

● **Assumption: Innovation is approx. Gaussian**

$$\epsilon(n) := y(n) - \hat{y}(n|n-1)$$

$$\hat{y}(n|n-1) = \mathbf{h}^T(n)\hat{\mathbf{x}}(n|n-1)$$

***Sinopoli et al, 2004; *Ribeiro et al, 2006; *K.Y. You, L.H. Xie, S.L. Sun & W.D. Xiao, IFAC Congress, 2008**

Quantized filtering

- **Model:**

$$\mathbf{x}(n) = \mathbf{A}(n)\mathbf{x}(n-1) + \mathbf{w}(n)$$
$$y(n) = \mathbf{h}^T(n)\mathbf{x}(n) + v(n)$$

- **Quantized the innovation:**

$$\epsilon(n) := y(n) - \hat{y}(n|n-1)$$

$$b(n) := \begin{cases} z_N, & \bar{z}_N < \epsilon(n) \\ z_{N-1}, & \bar{z}_{N-1} < \epsilon(n) \leq \bar{z}_N \\ \vdots & \vdots \\ z_1, & \bar{z}_1 < \epsilon(n) \leq \bar{z}_2 \\ 0, & -\bar{z}_1 < \epsilon(n) \leq \bar{z}_1 \\ -z_1, & -\bar{z}_2 < \epsilon(n) \leq -\bar{z}_1 \\ \vdots & \vdots \\ -z_N, & \epsilon(n) \leq -\bar{z}_N \end{cases}$$

- **Kalman filter with quantized innovation:**

$$\hat{\mathbf{x}}(n|n-1) := E[\mathbf{x}(n)|\mathbf{b}_{0:n-1}] = \mathbf{A}(n)\hat{\mathbf{x}}(n-1|n-1)$$
$$\hat{y}(n|n-1) := E[y(n)|\mathbf{b}_{0:n-1}] = \mathbf{h}^T(n)\hat{\mathbf{x}}(n|n-1)$$

$$\mathbf{P}(n|n-1) = \mathbf{A}(n)\mathbf{P}(n-1|n-1)\mathbf{A}^T(n) + \mathbf{W}(n)$$

$$\hat{\mathbf{x}}(n|n) = \hat{\mathbf{x}}(n|n-1) + \frac{f_N(n)\mathbf{P}(n|n-1)\mathbf{h}(n)}{\sqrt{\mathbf{h}^T(n)\mathbf{P}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)}}$$

$$\mathbf{P}(n|n) = \mathbf{P}(n|n-1) - 2\sum_{k=1}^{N} \frac{[\phi(z_k) - \phi(z_{k+1})]^2}{\alpha_{z_k} - \alpha_{z_{k+1}}} \times \frac{\mathbf{P}(n|n-1)\mathbf{h}(n)\mathbf{h}^T(n)\mathbf{P}(n|n-1)}{\mathbf{h}^T(n)\mathbf{P}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)}$$

$$f_N(n) = \sum_{k=0}^{N} Sgn(b(n)) I_{\{k\}}(b(n)) \frac{\phi(z_k) - \phi(z_{k+1})}{\alpha_{z_k} - \alpha_{z_{k+1}}}$$

* K. You, L. Xie, S. Sun & W. Xiao, IFAC, 2008

Kalman filter with 1-level quantizer

- **Model:** $\mathbf{x}(n) = \mathbf{A}(n)\mathbf{x}(n-1) + \mathbf{w}(n)$

$$y(n) = \mathbf{h}^T(n)\mathbf{x}(n) + v(n)$$

$$\hat{\mathbf{x}}(n|n) := E[\mathbf{x}(n)|\mathbf{b}_{0:n}] = \int_{\mathbb{R}^p} \mathbf{x}(n)p[\mathbf{x}(n)|\mathbf{b}_{0:n}]d\mathbf{x}(n)$$

- **Quantizer:**

$$b(n) := \begin{cases} z_1, & \bar{z}_1 < \epsilon(n) \\ 0, & -\bar{z}_1 < \epsilon(n) \le \bar{z}_1 \\ -z_1, & \epsilon(k) \le -\bar{z}_1 \end{cases}$$

- **Kalman filter:**

$$\hat{\mathbf{x}}(n|n) = \hat{\mathbf{x}}(n|n-1) + f_1(n) \times \frac{\mathbf{P}(n|n-1)\mathbf{h}(n)}{\sqrt{\mathbf{h}^T(n)\mathbf{P}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)}}$$

$$\mathbf{P}(n|n) = \mathbf{P}(n|n-1) - \frac{2\phi^2(n)}{\alpha_{z_1}} \times \frac{\mathbf{P}(n|n-1)\mathbf{h}(n)\mathbf{h}^T(n)\mathbf{P}(n|n-1)}{\mathbf{h}^T(n)\mathbf{P}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)}$$

$$\frac{1}{\sqrt{2\pi}} exp(-\frac{x^2}{2})$$

$$f_1(n) := \frac{\phi(z_1)}{\alpha_{z_1}} Sgn(b(n))$$

$$\int_{z_1}^{\infty} \phi(x)dx$$

*K.Y. You, L.H. Xie, S.L. Sun & W.D. Xiao, IFAC Congress, 2008

*** For the case with1-bit transmission, a better performance is obtained compared with that of the sign-innovation filter given (Ribeiro, 2006)**

Adaptive control with binary-valued data

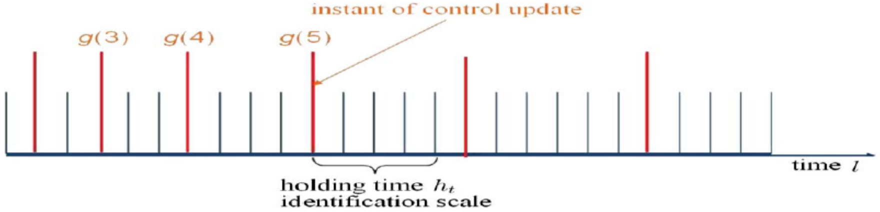
- Adaptive control with empirical measure based identification
- Adaptive control with recursive projection identification

Adaptive control with binary-valued data

- **Model:** $\begin{cases} y(k) = \varphi_k^T\theta + d_k \\ s(k) = I_{\{y(k)\le C\}} \end{cases}$

- **Goal:** $y(k) \to y^*$: y^* is m-periodic signal

- **Two-scale adaptive control:**



Estimation:

$$\xi_t = \left(\frac{1}{t-1} \sum_{l=g(t-1)}^{g(t)-1} \mathbf{s}(l) \right) \Bigg|_{\varepsilon_2}$$

$$\widehat{\theta}\left(g(t)\right) = \Phi\left(g(t-1)\right)^{-1} \left(C - F^{-1}(\xi_t)\right) \mathbb{1}_m$$

$$\widehat{\Theta}(g(t)) = \begin{cases} \widehat{\Theta}\left(g(t)\right), & \text{if } \left|\det\left(\Theta\left(g(t)\right)\right)\right| > \varepsilon_1; \\ \widehat{\Theta}\left(g(t-1)\right), & \text{otherwise.} \end{cases}$$

At time $l = g(t)+1, \ldots, g(t+1)-1$, let

$$\widehat{\Theta}(l) = \widehat{\Theta}\left(g(t)\right).$$

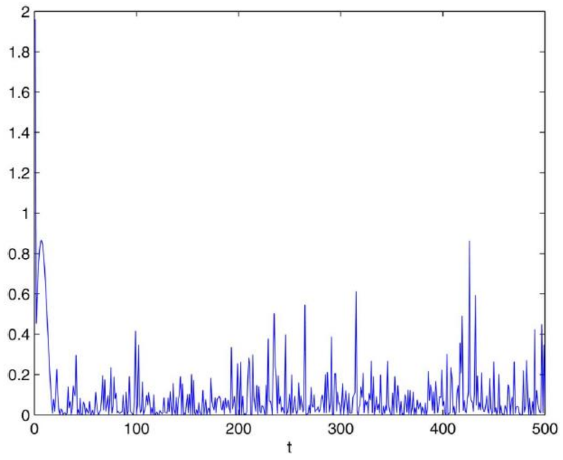
where $\varepsilon_1 = \varepsilon_0/2$ and $\varepsilon_2 = \left(1 - F\left(nC + M\|Y\|_2/\varepsilon_1\right)\right)/2$

$$g(t) = \frac{t(t+1)}{2}, \quad x|_\varepsilon = xI_{\{\varepsilon \le x \le 1-\varepsilon\}}. \qquad \begin{aligned} \mathbf{s}(l) &= \left[s(lm), \ldots, s\left((l-1)m+1\right)\right]^T \\ \Phi(l) &= \left[\phi(lm), \ldots, \phi\left((l-1)m+1\right)\right]^T \end{aligned}$$

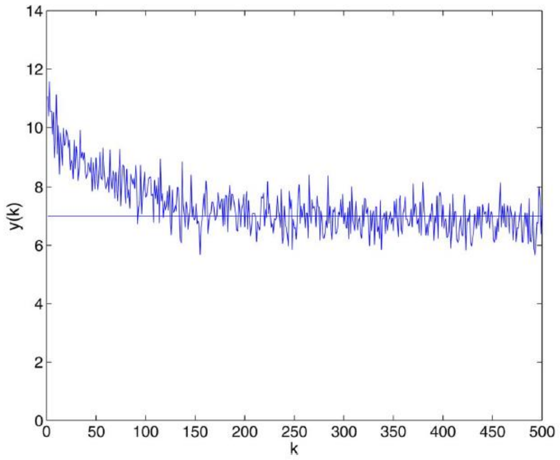
Control: $\Phi(g(t)) = Y\widehat{\Theta}\left(g(t)\right)^{-1}$

where $Y = T([y_m^*, \cdots, y_1^*])$

Property: asymptotically efficient estimate;
mean square convergence rate $O\left(\frac{1}{\sqrt{t}}\right)$;
asymptotically optimal control;



Trajectory of $t\left(\hat{\theta} - \theta\right)^2$ in one holding time



System output with tracking target $y^* = 7$

* Y. Zhao, J. Guo & J. F. Zhang, IEEE TAC, 2013

Adaptive control with binary-valued data

- **Adaptive control with time-varying threshold**

Estimation:

$$\hat{\theta}(t+1) = \Pi_{\Omega}\left(\hat{\theta}(t) - \frac{\alpha}{t}\Phi^T(t)\left(\mathcal{F}(C\vec{1} - \Phi^T(t)\hat{\theta}(t)) - s(t)\right)\right)$$

Control:

$$\Phi(t+1) = Y\widehat{\Theta}(t+1)^{-1} I_{\{\lambda_{\min}\left(\widehat{\Theta}(t+1)\widehat{\Theta}^T(t+1)\right)\geq\varepsilon_0\}} + \frac{Y}{\sqrt{\varepsilon_0}} I_{\{\lambda_{\min}\left(\widehat{\Theta}(t+1)\widehat{\Theta}^T(t+1)\right)<\varepsilon_0\}}$$

where $\mathcal{F}(x) = \left(F\left(x(1)\right), \cdots, F\left(x(n)\right)\right)^T$

Property:

The designed input satisfy $\|\Phi_t\| = \|\Phi_t^T\| \leq M$,

and

$$\Phi_t^T \Phi_t \geq \delta I_L, \text{ for } t = 1, 2, \ldots.$$

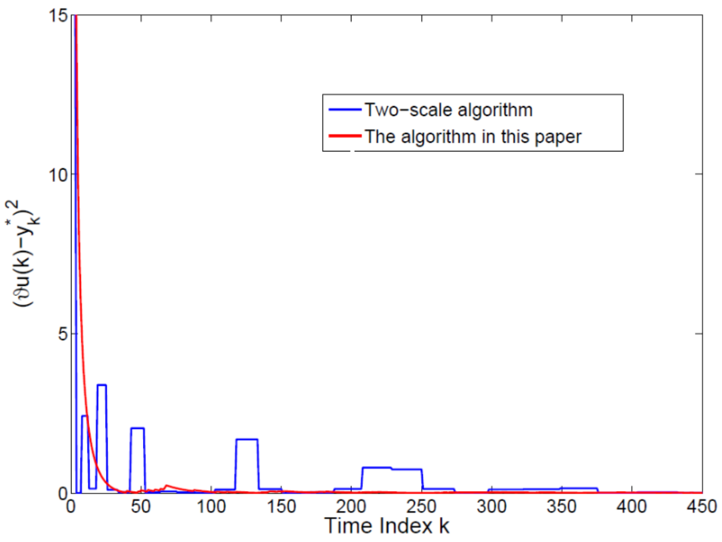
Mean square convergence rate

$$\mathrm{E}(\tilde{\theta}_t^T \theta_t) = O\left(\frac{1}{t}\right)$$

if $f(C - \phi^T(l)\vartheta) \geq \underline{f} > \frac{1}{2\alpha\delta}$

Asymptotically optimal control;

$$\lim_{t\to\infty} E(Y_t - Y^*)^T(Y_t - Y^*) = L\sigma^2$$



* T. Wang, M. Hu & Y. Zhao, IEEE SMC, 2019

Consensus with quantized data

- Output feedback consensus
- Consensus with quantized inputs

Output feedback consensus with finite-level quantization

Model: $\begin{cases} x_i(t+1) = Ax_i(t) + Bu_i(t) \\ y_i(t) = Cx_i(t) \end{cases}$

Communication network: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$

Communication protocol set:

$$\mathcal{H}(\rho, L_G) = \left\{ \begin{array}{l} H(\gamma, \alpha, \alpha_u, L, L_u, G) = \{H_{ji} = (\Theta_j, \Psi_{ji}), i = 1, \cdots, N, j \in \mathcal{N}_i\}, \\ \gamma \in (0, \rho), \alpha, \alpha_u \in (0,1], L, L_u \in \mathbb{N}, G \in \mathcal{B}_{L_G}^{n\times p} \end{array} \right\}$$

Quantizer:

$$Q_{p,M}(y) = \begin{cases} kp, kp - p/2 \le y < kp + p/2, k = 0,1,\cdots, M-1, \\ Mp, y \ge Mp - p/2, \\ -Q_{p,M}(-y), y < -p/2. \end{cases}$$

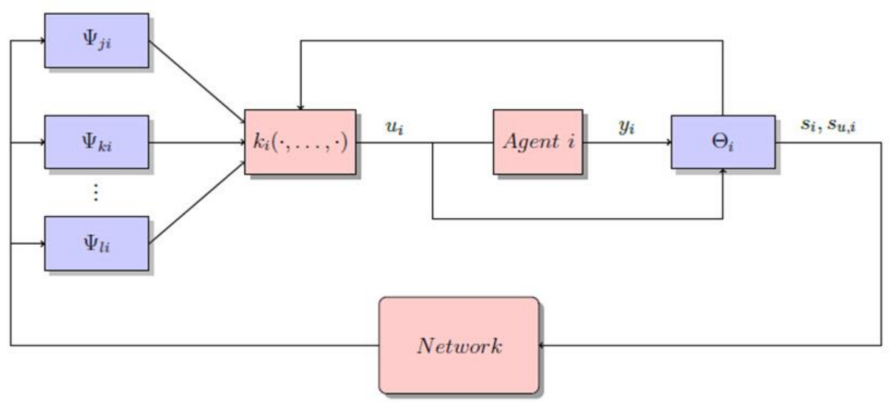
Control protocol set:

$$\mathcal{U}(L_K) = \{U(\mathrm{K}), K \in \mathcal{B}_{L_k}^{m\times n}\},$$

Control goal: locally/globally consensus

For any $C_1, C_2, C_3, \exists\, H \in \mathcal{H}, U \in \mathcal{U}$, s. t., for any $x_i(0) \in \mathcal{B}_{C_1}^n, \hat{x}_{i0} \in \mathcal{B}_{C_2}^n, \hat{u}_{i0} \in \mathcal{B}_{C_3}^n$, the closed-loop dynamic system $(A, B, C, \mathcal{G})$ achieves:

$$\lim_{t\to\infty} \left(x_j(t) - \hat{x}_{ji}(t)\right) = \mathbf{0}, \lim_{t\to\infty} \left(x_j(t) - x_i(t)\right) = \mathbf{0}.$$



$$\Theta_j = \begin{cases} \hat{x}_j(0) = \hat{x}_{j0}, \hat{u}_j(0) = \hat{u}_{j0}, \\ s_j(t) = Q_{\alpha,L}\left(y_j(t-1) - C\hat{x}_j(t-1)/\gamma^{t-1}\right), \\ \hat{x}_j(t) = A\hat{x}_j(t-1) + \gamma^{t-1} G s_j(t) + B\hat{u}_j(t-1), \\ s_{u,j}(t) = Q_{\alpha,L}\left(u_j(t) - \hat{u}_j(t-1)/\gamma^{t-1}\right), \\ \hat{u}_j(t) = \hat{u}_j(t-1) + \gamma^{t-1} s_{u,j}(t), \end{cases}$$

$$\Psi_{ji} = \begin{cases} \hat{x}_{ji}(0) = \hat{x}_{j0}, \hat{u}_{ji}(0) = \hat{u}_{j0}, \\ \hat{x}_{ji}(t) = A\hat{x}_{ji}(t-1) + \gamma^{t-1} G s_j(t) + B\hat{u}_{ji}(t-1), \\ \hat{u}_{ji}(t) = \hat{u}_{ji}(t-1) + \gamma^{t-1} s_{u,j}(t). \end{cases}$$

$$U(K) = \left\{ u_i(t) = K \sum_{j\in\mathcal{N}_i} a_{ij}(\hat{x}_{ji}(t) - \hat{x}_i(t)), j = 1, \cdots, N \right\}$$

Output feedback consensus with finite-level quantization

Assumption:

A1) There exists K such that the eigenvalues of $A-\lambda_i(\mathcal{L})BK, i=2,\cdots,N$ are all inside the open unit disk of the complex plane.

A2) (A,C) is detectable.

Conclusion:

- **Sufficiency:**

$$\textbf{A1)+A2)}$$
$$\Downarrow$$
$$(A,B,C,\mathcal{G}) \text{ is locally consensus for } \mathcal{H}(1,+\infty) \;\&\; \mathcal{U}(+\infty);$$
$$+\text{Uniform boundedness of the quantization errors;}$$

- **Necessity:**

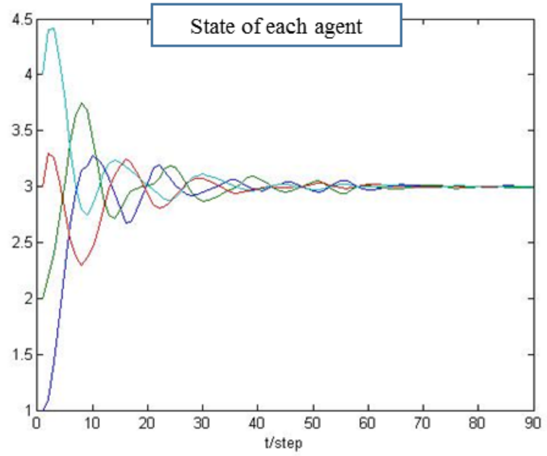
1) $(A,B,C,\mathcal{G})$ is locally consensus for $\mathcal{H}(\rho, L_G)$and $\mathcal{U}(L_k)$ with $\rho\in(0,1), L_G>0, L_K>0$;
+Uniform boundedness of the quantization errors;

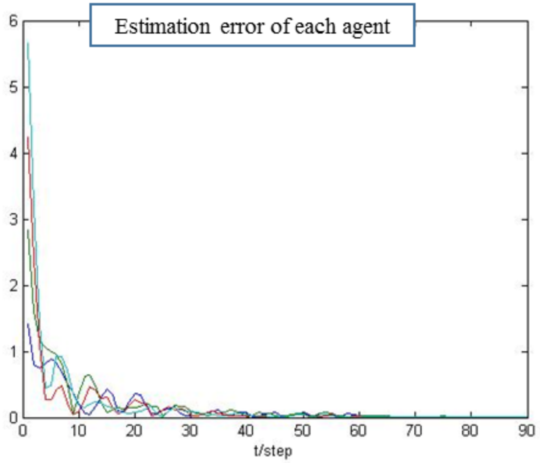
$$\Downarrow$$
$$\textbf{A1)+A2)}$$

2) $(A,B,C,\mathcal{G})$ is globally consensus for $\mathcal{H}(1,+\infty)$ and $\mathcal{U}(+\infty)$;
+Uniform boundedness of the quantization errors;

$$\Downarrow$$
$$\textbf{A1)+A2)}$$

- $(A,B,C,\mathcal{G})$ is globally consensus for $\mathcal{H}(1,+\infty)$ and $\mathcal{U}(+\infty)$ with accurate communication $\Leftrightarrow$ **A1)+A2)**





*** Y. Meng, T. Li & J.F. Zhang, IEEE TAC, 2017**

Highlight:

1) Unstable and high-order system + unmeasurable states

2) Dual goals of communication (finite data rate) and control

3) Sufficiency + necessity

Other case:

Case 1: switching and frequently connected commun. network

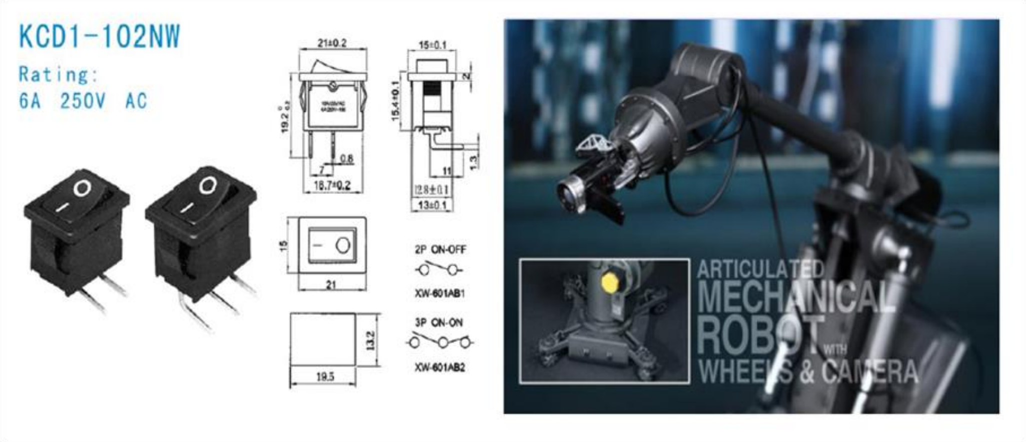
*** Y. Meng, T. Li & J.F. Zhang, SICON, 2017**

Case 2: jointly connected communication network

*** Y. Meng, T. Li & J.F. Zhang, IJRNC, 2015**

Consensus with quantized inputs

Example of input sets with limited precision:
Switch and mechanical arm



Model: $x_i(t+1) = x_i(t) + u_i(t)$

Communication network: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$

Quantizer (finite level): $Q_L(y)$

The input set with limited precision:

$$\mathcal{U} = \{\pm\mu_k, k = 1, \cdots, L\} \cup \{0\}$$

Control goal: practical consensus

$$\overline{\lim_{t\to\infty}} |x_j(t) - x_i(t)| \le \varepsilon.$$

Kalman pointed out that
the input set with limited precision$\Longrightarrow$ **limit cycles** or **chaos.**

Highlight:
1) Single agent $\Rightarrow$ multi-agent
2) limitless precision $\Rightarrow$ limited precision
3) Sufficiency + necessity

Control protocol set:

$$\mathcal{C} = \{U(f(\cdot), h) | U = \{u_i(t), i = 1, \cdots, N\}, Q_L(\cdot): \mathbb{R} \to \mathcal{U}, h \in (0, 2/\lambda_N)\}$$

where $u_i(t) = Q\left(h \sum_{i=1}^{N} a_{ij}\left(x_j(t) - x_i(t)\right)\right)$ and $\lambda_N = \max \lambda_i(\mathcal{L})$

Case 1: logarithmically distributed input sets

i.e., $\mathcal{U} = \{\pm\mu_k, k = 1, 2, \cdots\} \cup \{0\}$ with $\mu_k = \rho^k \mu_0, \rho \in (0,1)$

$$Q_L(y) = \begin{cases} \rho^l \mu_0, \frac{\rho^l \mu_0}{1+\beta} \le y < \frac{\rho^l \mu_0}{1-\beta}, l = 1, 2, \cdots, \\ 0, y = 0, \\ -Q_L(-y), y < 0. \end{cases}$$ and $\beta = \frac{1-\rho}{1+\rho}$

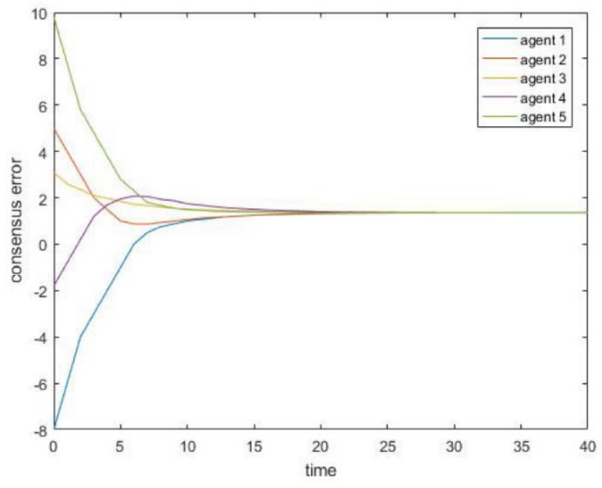
Consensus with quantized inputs

Conclusion:

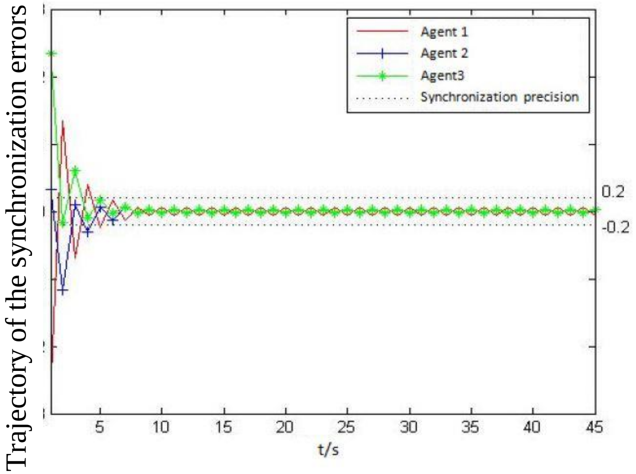
1) $\mathcal{G}$ is connected;
2) $|x_i(0)| \le C_x$;
3) $\beta < \lambda_2 / \lambda_N$;

$\Downarrow$

consensus exponentially



Trajectory of the synchronization errors



* Y. Meng, Z. Wang, Assembly Automation, 2016

Case 2: uniformly distributed input sets

i.e., $\mathcal{U} = \{\pm\mu_k, k = 1, \cdots L\} \cup \{0\}$ with $\mu_k = k\omega, \rho \in (0,1)$

$$Q_L(y) = \begin{cases} k\omega, k\omega - \omega/2 \le y < k\omega + \omega/2, k = 0,1,\cdots, L-1, \\ L\omega, y \ge L\omega - \omega/2, \\ -Q_L(-y), y < -\omega/2. \end{cases}$$

Assumption:

A1) $\mathcal{G}$ is connected;

A2) There is known constant C_x such that $|x_i(0)| \le C_x$;

Conclusion:

- **Sufficiency:** under A1)-A2) and $L \ge \frac{8d^*\sqrt{N}C_x}{\varepsilon(\lambda_2+\lambda_N)} + \frac{\sqrt{N}d^*}{\lambda_2} - \frac{1}{2}$,

$$\omega < \frac{4\varepsilon\lambda_2}{\sqrt{N}(\lambda_2+\lambda_N)}$$

$$\Downarrow$$

practical consensus with ε

- **Necessity:** under A1)-A2) and practical consensus with ε,

$$\overline{\lim_{t\to\infty}} \left\| X(t) - \bar{x}(t)\vec{1} \right\| \le \varepsilon$$

$$\Downarrow$$

$$\omega < \frac{4\varepsilon\lambda_2}{\sqrt{N}(\lambda_2+\lambda_N)}$$

Contents

- Why do we consider quantized systems
- What are the fundamental problems
- What are the recent progresses:
  filtering, identification, control
- **What are the problems worth studying**

Identification and adaptive control with quantized data

- **Parameter identification with quantized data**
  - ✓ Bisection method & param. decoupling for noise-free/bounded noises
  - ✓ Likelihood method for the case with stochastic noises
  - ✓ Expectation maximization method
  - ✓ Empirical measure method with/without truncation
  - ✓ Recursive projection algorithm:
    - * Stochastic approximation: scalar step, known distr. function
    - * Sign-error: scalar step, time-varying threshold, unknown distr. funct.
    - * CRLB based: matrix step depending on estimate, asymp. efficiency
    - * Quasi-Newton: matrix step, weak persistent excitation

- **Stochastic approximation and state estimation with quantized data**

- **Adaptive control with binary-valued data**
  - ✓ Adaptive control with empirical measure based ident.
  - ✓ Adaptive control with recursive projection ident.

- **Consensus with quantized data**
  - ✓ Output feedback consensus
  - ✓ Consensus with quantized inputs

- Persistent excitation, periodic input, scaled periodic input, weak excitation, ……
- Convergence, convergence rate, asymptotic efficiency, asymptotic optimality, ……

- **Open questions**
  - ✓ Asymptotically optimal algorithm
  - ✓ State space model
  - ✓ MIMO systems

Feature and difficulty on quantized system research

Research on quantized systems is systematic

- **Wide-range:** estimation, identification, control, et al
- **General framework:** a research framework can be established parraleling to the one with precise output
- **Significancy:** essentially reduce the requirements on measurements

Difficulty in modelling and control of quantized systems

- **Algorithm design:** less available information, strong nonlinearity
- **Theoretical analysis:** the matrix is not independent, non-exchangeable and contains unknown parameters, etc.

- **Basic question:**

  In order to reach a desired modelling or control goal, how much information do we really need ?

- **It is involved in unified design of control and communication, and needs to develop "control-based information theory"**

- **It is a complex function in terms of task, constraint, complexity**
  - ✓**Task:** modeling, identification, or control, … …
  - ✓**Constraint:** dynamic, measurement, cost, time, bandwidth, processing, … …
  - ✓**Complexity:** Computation, implementation, analysis, ……

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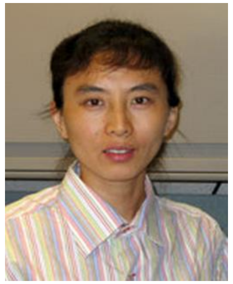
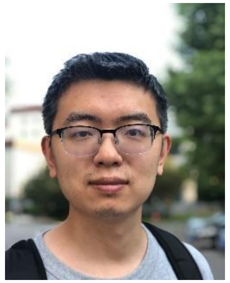

G.L. Kang


Y.L. Zhao


T. Li


Q. Zhang


J. Guo


W.J. Bi


Y. Meng


T. Wang


H. Zhang


L.D. Jing


Y. Wang

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